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NUMERICAL SOLUTION BY FINITE ELEMENT METHOD FOR TIME CAPUTO-FABRIZIO FRACTIONAL PARTIAL DIFFUSION EQUATION

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Abstract. The foundation of this study lies in the fractional order derivative definition pioneered by Caputo and Fabrizio, with distinct representations for temporal and spatial variables. Our focus is on presenting a finite element scheme to address the time-fractional diffusion equation. This approach contrasts with traditional finite difference methods and variational solutions. We rigorously establish the stability and convergence order of our proposed method. To validate our theoretical assertions, we provide a numerical example demonstrating its efficacy.

Keywords: Finite element method, partial differential equations, fractional order derivative, stability analysis and convergence, error estimates.

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1 Introduction

Traditionally, calculus deals with integer orders of derivatives and integrals. However, the curiosity about what would happen if we extended this concept to non-integer or fractional orders led to the development of fractional calculus. This seemingly simple question opened up a rich and important new area of mathematical theory with applications in various fields such as chemistry, physics, mathematical biology, fractal media, electromagnetic, statistical mechanics, and many other fields. For almost 300 years, researchers have delved into and expanded the field of fractional calculus. Initially confined to pure mathematics for over two centuries, recent decades have seen its recognition and utilization analytically and numerically in various natural contexts and practical applications. For instance, Chaurasia et al. (2012), Manafian et al. (2015), Povstenko (2014); Klekot et al. (2016), Salim et al. (2009); Velieva & Agamalieva (2017), Stern et al. (2014) and Yusubov (2015) studied the existence of analytical solution of fractional differential equations. Meanwhile, others used various numerical methods, such as Li et al. (2009) and Meerschaert et al. (2004) used the finite difference method, Hejazi et al. (2013) and

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Zhuang et al. (2014) used the finite volume method, Li et al. (2009), Li et al. (2009) and Zheng et al. (2015) used the finite element method and Shahriari et al. (2020) used spectral method and presented a computational algorithm to solve the one dimensional fractional Dirac operator.

This sparked the curiosity of mathematicians, who began exploring the possibilities of changing orders, kernels, and neighborhoods based on their understanding of ordinary integral and differential operators. Since fractional derivatives possess fewer properties, mathematicians worked on introducing various fractional derivatives to overcome limitations and challenges. Examples of these derivatives include Riemann-Liouville, Grunwald-Letnikov, and Erdelyi-Kober, among others. Some of these derivatives have proven to be powerful tools in exploring the complex dynamics of real-world phenomena and introducing fractional order operators into disciplines like physics, chemistry, engineering, biology, and medicine. However, other definitions are less utilized and still await further exploration.One notable derivative was introduced by Caputo in 1967, who reformulated the definition of the Riemann-Liouville fractional derivative by swapping the order of the ordinary derivative with the fractional integral operator to create his new definition. While it may not be effective for non-integer order derivatives, Caputo's derivative is particularly useful in solving fractional differential equations, especially since it doesn't require fractional order initial conditions. These equations arise in various scientific and engineering applications, such as viscoelasticity, diffusion, and control systems. Caputo et al. (2015) further improved this derivative by changing its kernel to an exponential one to overcome its singularity. This derivative garnered significant interest due to its dual representations of temporal and spatial variables, as well as the absence of a singular kernel. Below, we highlight some works that show the elegance and utility of this derivative. Losada et al. (2015) defined the fractional integral corresponding to this fractional order derivative. Nieto et al. (2015) presented a numerical solution of the RLC circuit model which uses the fractional order derivative without a singular kernel. Atangana (2016) presented some useful and interesting tools about the definition and applied them back to the nonlinear reaction-diffusion of the Fisher equation. Gómez et al. (2016) presented another alternative representation of the diffusion and diffusion-advection equations using the definition of the Caputo and Fabrizio to approximate the spatial and time derivatives. Alqahtani et al. (2016) proposed a numerical approximation for the space-time of Caputo-Fabrizio fractional order derivative and applied it to the equation of groundwater pollution. Cheng et al. (2017) used it to solve the equation of fractional Cattaneo based on this fractional order derivative present a second-order Crank-Nicklson scheme, and Liu et al. (2018) proposed a second-order finite difference scheme to solve the quasilinear time parabolic equation with Caputo-Fabrizio fractional order. Can et al. (2020) and Jafari et al. (2023) give recently novel numerical methods for solving special fuzzy and non-linear fractional differential equations with various kernels. Nevertheless, to our best knowledge, finite element methods for solving partial differential equations with fractional order derivatives based on this fractional order derivative have not been reported yet.

Boutiba et al. (2022) assumed in two different representations for the temporal and the spatial variables to solve the time fractional partial differential equations, based on the Riemann-Liouville fractional derivative giving stability and convergence order of the finite element method and proving that the semi-discretization is unconditionally stable. And Boutiba et al. (2023) solved the fractional space-time diffusion equation over finite fields by replacing the first temporal derivative with the Caputo-Fabrizio fractional derivative and the second spatial derivative with the Riemann-Liouville fractional derivative.

In this paper, we extend previous results involving Caputo and Fabrizio's time-fractional derivative presuming two different representations for the temporal and spatial variables with the Laplacian operator to solve the time-fractional diffusion equation with non-homogenous initial and limit conditions using finite difference and element schemes to establish stability and convergence order of the method.

In this paper, we consider the following fractional time diffusion equation

$$\begin{cases} {}_0^{CF}\mathcal{D}_t^{\gamma}\mathcal{U}(x,t) - \Delta\mathcal{U}(x,t) = f(x,t), & (x,t) \in [a,b] \times [0,T], \\ \mathcal{U}(x,0) = \psi(x), & x \in [a,b] = \Omega, \\ \mathcal{U}(a,t) = \mathcal{U}_a(t), \quad \mathcal{U}(b,t) = \mathcal{U}_b(t), & t \in [0,T] = I, \end{cases} \tag{1}$$

where $b > a > 0$; $T > 0$; $f : \Omega \times I \to \mathbb{R}$; $\psi : \Omega \to \mathbb{R}$; $\mathcal{U}_a, \mathcal{U}_b : I \to \mathbb{R}$ are given functions; Δ is the Laplacian operator and ${}_0^{CF}\mathcal{D}_t^{\gamma}\mathcal{U}(x,t)$ is the Caputo-Fabrizio fractional order derivative given by Caputo et al. (2015) and Losada et al. (2015) for $0 < \gamma < 1$,

$${}_0^{CF}\mathcal{D}_t^{\gamma}\mathcal{U}(x,t) = \frac{1}{1-\gamma}\int_a^t \frac{\partial\mathcal{U}(x,\xi)}{\partial\xi} e^{\left[-\gamma\dfrac{t-\xi}{1-\gamma}\right]} d\xi. \tag{2}$$

Our goal is to introduce a finite element method for solving this fractional time diffusion equation. We'll establish unconditionally stable results and derive some a priori estimates. This work contains four sections. In Section 2, we discuss the time discretization of the problem (1), examine the existence and uniqueness of a weak solution, investigate its unconditional stability, and provide error estimates for the semi-discrete scheme. In Section 3, we present the fully discrete scheme used and carry out error estimates for the problem (1). Finally in Section 4, we provide a numerical example to validate our theoretical findings.

2 Time discretization

In this section, we present the semi-discrete variational form of the fractional time diffusion equation. We then discuss the existence and uniqueness of the solution, followed by stability and convergence analysis.

2.1 Finite difference scheme

First, to approximate the fractional time derivative, we need to discretize the space-time as $t_j = j\Delta t, j = 0, 1, \cdots, J$ where $\Delta t = \dfrac{T}{J}$ is the time step. Hence the Caputo et al. (2015) fractional time derivative (2) is estimated as follows,

$$\begin{aligned} {}_0^{CF}\mathcal{D}_t^{\gamma}\mathcal{U}(x,t_{j+1}) &= \frac{1}{1-\gamma}\int_0^{t_{j+1}} \frac{\partial\mathcal{U}(x,\xi)}{\partial\xi} e^{\left[-\gamma\dfrac{t_{j+1}-\xi}{1-\gamma}\right]} d\xi \\ &= \frac{1}{1-\gamma}\sum_{k=0}^{j}\int_{t_k}^{t_{k+1}} \frac{\partial\mathcal{U}(x,\xi)}{\partial\xi} e^{\left[-\gamma\dfrac{t_{j+1}-\xi}{1-\gamma}\right]} d\xi. \end{aligned} \tag{3}$$

Since,

$$\frac{\partial\mathcal{U}(x,\xi)}{\partial\xi} = \frac{\mathcal{U}(x,t_{k+1}) - \mathcal{U}(x,t_k)}{\Delta t} + (\xi - t_k)\mathcal{U}_{tt}(x,c_k)$$

with $c_k \in (t_k, t_{k+1})$. Then, using (3), the Caputo-Fabrizio fractional time derivative becomes

$$
\begin{aligned}
&{}_0^{CF}\mathcal{D}_t^{\gamma}\mathcal{U}(x,t_{j+1}) = \\
&= \frac{1}{1-\gamma}\sum_{k=0}^{j}\int_{t_k}^{t_{k+1}}\left[\frac{\mathcal{U}(x,t_{k+1})-\mathcal{U}(x,t_k)}{\Delta t}+(\xi-t_k)\mathcal{U}_{tt}(x,c_k)\right]e^{\left[-\gamma\frac{t_{j+1}-\xi}{1-\gamma}\right]}d\xi, \\
&= \frac{1}{1-\gamma}\sum_{k=0}^{j}\frac{\mathcal{U}(x,t_{j-k+1})-\mathcal{U}(x,t_{j-k})}{\Delta t}\int_{t_k}^{t_{k+1}}e^{\left[-\gamma\frac{\xi}{1-\gamma}\right]}d\xi + E_{\Delta t}, \\
&= \frac{1}{\gamma}\sum_{k=0}^{j}\frac{\mathcal{U}(x,t_{j+1-k})-\mathcal{U}(x,t_{j-k})}{\Delta t}\left(e^{\left[-\gamma\frac{t_k}{1-\gamma}\right]}-e^{\left[-\gamma\frac{t_{k+1}}{1-\gamma}\right]}\right)+E_{\Delta t}.
\end{aligned}
$$

Then,

$$
{}_0^{CF}\mathcal{D}_t^{\gamma}\mathcal{U}(x,t_{j+1}) = \frac{1}{\gamma}\sum_{k=0}^{j}\mathcal{M}_k\mathcal{U}_t(x,t_{j-k+1})+E_{\Delta t}, \tag{4}
$$

where

$$
\begin{cases}
\mathcal{M}_k = e^{\left[-\gamma\frac{k\Delta t}{1-\gamma}\right]}-e^{\left[-\gamma\frac{(k+1)\Delta t}{1-\gamma}\right]}, \\
\mathcal{U}_t(x,t_{j-k+1}) = \dfrac{\mathcal{U}(x,t_{j-k+1})-\mathcal{U}(x,t_{j-k})}{\Delta t},
\end{cases} \tag{5}
$$

and $E_{\Delta t}$ is the truncation error given by

$$
E_{\Delta t} = {}_0^{CF}\mathcal{D}_t^{\gamma}\mathcal{U}(x,t_{j+1}) - \frac{1}{\gamma}\sum_{k=0}^{j}\mathcal{M}_k\mathcal{U}_t(x,t_{j-k+1}), \tag{6}
$$

$$
= \frac{1}{1-\gamma}\sum_{0}^{j}\int_{t_k}^{t_{k+1}}(\xi-t_k)\mathcal{U}_{tt}(x,c_k)e^{\left[-\gamma\frac{t_{j+1}-\xi}{1-\gamma}\right]}d\xi. \tag{7}
$$

Suppose that $\mathcal{U}(t)\in C^2([0,t_k];\mathbb{R})$, then we have

$$
|E_{\Delta t}| \le C_{\mathcal{U},\gamma}(\Delta t)^2,
$$

such that

$$
C_{\mathcal{U},\gamma} = \frac{1}{\gamma}\max_{1\le k\le j+1}\|\mathcal{U}_{tt}(x,c_k)\|\, e^{\left[\frac{2\gamma}{1-\gamma}\right]}.
$$

Now, for the brevity's sake, we set

$$
Q_t^{\gamma}\mathcal{U}(x,t_{j+1}) = \frac{1}{\gamma}\sum_{k=0}^{j}\mathcal{M}_k\mathcal{U}_t(x,t_{j-k+1}).
$$

Consequently, (3) leads to

$$
{}_0^{CF}\mathcal{D}_t^{\gamma}\mathcal{U}(x,t_{j+1}) = Q_t^{\gamma}\mathcal{U}(x,t_{j+1})+E_{\Delta t}.
$$

Then, we will use $Q_t^{\gamma}(x,t_{j+1})$ as an approximation of the time Caputo-Fabrizio fractional order derivative which leads us to the next finite difference scheme of the problem (1),

$$
Q_t^{\gamma}\mathcal{U}^{j+1}-\Delta\mathcal{U}^{j+1}=f^{j+1},\quad j=0,1,\cdots,J-1.
$$

After some adjustment, we obtain the following scheme

$$\mathcal{U}^{j+1} - r\Delta\mathcal{U}^{j+1} = \sum_{k=0}^{j} \mathcal{N}_{j,k}\mathcal{U}^k + rf^{j+1},$$

in which $r = \dfrac{\gamma\Delta t}{\mathcal{M}_0}$ and

$$\mathcal{M}_0.\mathcal{N}_{j,k} = \begin{cases} \mathcal{M}_j, & k = 0, \\ \mathcal{M}_{j-k} - \mathcal{M}_{j-k+1}, & 1 \leq k \leq j. \end{cases}$$

2.2 Existence and uniqueness of the variational solution

Our goal is to introduce the variational formulation of problem (1) and derive the existence and uniqueness of the solution, we need to define the next functional spaces with their norms

$$H^1(\Omega) = \left\{ w \in L^2(\Omega), \frac{dw}{dx} \in L^2(\Omega) \right\},$$
$$H_0^1(\Omega) = \left\{ w \in H^1(\Omega), w(a) = w(b) = 0 \right\},$$
$$H^m(\Omega) = \left\{ w \in L^2(\Omega), \frac{d^k w}{dx^k} \in L^2(\Omega) \quad \text{for all positive integer } k \leq m \right\},$$

where $L^2(\Omega)$ is the space of all measurable functions whose square is Lebesgue integral in Ω. The L^2 and H^1 inner product are defined respectively by

$$(\mathcal{U}, w) = \int_\Omega \mathcal{U}\mathcal{W} dx, \qquad (\mathcal{U}, \mathcal{W})_1 = (\mathcal{U}, \mathcal{W}) + (\frac{d\mathcal{U}}{dx}, \frac{d\mathcal{W}}{dx})$$

and their corresponding norms,

$$\|\mathcal{W}\|_0 = (\mathcal{W}, \mathcal{W})^{1/2}, \qquad \|\mathcal{W}\|_1 = (\mathcal{W}, \mathcal{W})_1^{1/2}.$$

Define the norm of the space $H^m(\Omega)$ by,

$$\|\mathcal{W}\|_m = \left(\sum_{k=0}^{m} \left\| \frac{d^k\mathcal{W}}{dx^k} \right\|_0^2 \right)^{1/2}.$$

Instead of using the standard H^1 norm, we are going to use the following norm

$$\|\mathcal{W}\|_1 = \left(\|\mathcal{W}\|_0^2 + b \left\| \frac{d\mathcal{U}}{dx} \right\|_0^2 \right)^{1/2}.$$

Problem (1) transformed into a semi-discrete variational problem which is given as

Find $\mathcal{U}^{j+1} \in H_0^1(\Omega)$ for $j = 0, 1, \cdots, J-1$, where $\mathcal{U}^{j+1}(x)$ is an approximation of $\mathcal{U}(x, t_{j+1})$ such that

$$\left(\mathcal{U}^{j+1}, \mathcal{W}\right) + r\left(\frac{\partial\mathcal{U}^{j+1}}{\partial x}, \frac{\partial\mathcal{W}}{\partial x}\right) = \sum_{k=0}^{j} \left(\mathcal{N}_{j,k}\mathcal{U}^k, \mathcal{W}\right) + r\left(f^{j+1}, \mathcal{W}\right). \tag{8}$$

We denote

$$\mathcal{B}\left(\mathcal{U}^{j+1}, \mathcal{W}\right) = \left(\mathcal{U}^{j+1}, \mathcal{W}\right) + r\left(\frac{\partial\mathcal{U}^{j+1}}{\partial x}, \frac{\partial\mathcal{W}}{\partial x}\right)$$

and

$$f_1 = \sum_{k=0}^{j} \mathcal{N}_{j,k}\mathcal{U}^k + rf^{j+1}, \quad \mathcal{F}(\mathcal{W}) = (f_1, \mathcal{W}).$$

Then, we get the variational form in its concise form as

$$\mathfrak{B}\left(\mathcal{U}^{j+1}, \mathcal{W}\right) = \mathcal{F}(\mathcal{W}).$$

Theorem 1. *For $0 < \gamma < 1$, and a sufficiently small step size Δt, there exists a unique solution $\mathcal{U}^{j+1}$ that satisfy* (8). *Furthermore, $\mathcal{U}^{j+1}$ satisfy*

$$\left\|\mathcal{U}^{j+1}\right\|_1 \leq C \left\|f_1\right\|_{-1}.$$

The existence and the uniqueness of the variational solution are assured by the very well-known Lax-Milgram Lemma. It consists to prove that the bilinear form $\mathfrak{B}$ is coercive over $H_0^1(\Omega)$, its continuity over $H_0^1(\Omega) \times H_0^1(\Omega)$, and the continuity of the linear form $\mathcal{F}$.

Proof. 1. The coercivity,

$$\begin{aligned} \mathfrak{B}(\mathcal{U}^{j+1},\mathcal{U}^{j+1}) &= (\mathcal{U}^{j+1},\mathcal{U}^{j+1}) + r\left(\frac{\partial \mathcal{U}^{j+1}}{\partial x}, \frac{\partial \mathcal{U}^{j+1}}{\partial x}\right) \\ &\geq \left\|\mathcal{U}^{j+1}\right\|_0^2 + r\left\|\frac{\partial \mathcal{U}^{j+1}}{\partial x}\right\|_0^2 \\ &\geq \left\|\mathcal{U}^{j+1}\right\|_1^2. \end{aligned}$$

2. The continuity,

$$\begin{aligned} \mathfrak{B}\left(\mathcal{U}^{j+1},\mathcal{W}\right) &= (\mathcal{U}^{j+1},\mathcal{W}) + r\left(\frac{\partial \mathcal{U}^{j+1}}{\partial x}, \frac{\partial \mathcal{W}}{\partial x}\right) \\ &\leq \left\|\mathcal{U}^{j+1}\right\|_0 \left\|\mathcal{W}\right\|_0 + r\left\|\frac{\partial \mathcal{U}^{j+1}}{\partial x}\right\|_0 \left\|\frac{\partial \mathcal{W}}{\partial x}\right\|_0 \\ &\leq \left\|\mathcal{U}^{j+1}\right\|_1 \left\|\mathcal{W}\right\|_1. \end{aligned}$$

Moreover, we can prove that the linear form $\mathcal{F}(.)$ is continuous over $H_0^1(\Omega)$ since $f_1 \in H_0^1(\Omega) \subset H^{-1}(\Omega)$. Then, we have that

$$|\mathcal{F}(\mathcal{W})| = \|f_1(\mathcal{W})\| = \|f_1\|_{-1} . \|\mathcal{W}\|_1$$

that achieves the proof. □

For convenience and without loss of generality, we consider $f \equiv 0$ in what follows.

2.3 Stability analysis and error estimate

Lemma 1. *Semi-discrete form* (8) *is unconditionally stable for a sufficiently small step size Δt and*

$$\left\|\mathcal{U}^{j+1}\right\|_1 \leq \left\|\mathcal{U}^0\right\|_0, for\ j = 0, \cdots, J-1.$$

Proof. We will use inductions to prove the result. First for $j = 0$ in (8), we have

$$(\mathcal{U}^1, \mathcal{W}) + r\left(\frac{\partial \mathcal{U}^1}{\partial x}, \frac{\partial x}{\partial x}\right) = \left(\mathcal{U}^0, \mathcal{W}\right), \quad \mathcal{W} \in H_0^1(\Omega).$$

Taking $\mathcal{W} = \mathcal{U}^1$, we obtain

$$\left\|\mathcal{U}^1\right\|_1 \leq \left\|\mathcal{U}^0\right\|_0.$$

Now, we suppose that

$$\left\|\mathcal{U}^k\right\|_1 \leq \left\|\mathcal{U}^0\right\|_0 \quad \text{for} \quad k = 0, \cdots, j. \tag{9}$$

We need to prove that

$$\left\|\mathcal{U}^{j+1}\right\|_1 \leq \left\|\mathcal{U}^0\right\|_0.$$

Taking $\mathcal{W}=\mathcal{U}^{j+1}$ in (8),

$$\left(\mathcal{U}^{j+1},\mathcal{U}^{j+1}\right)+r\left(\frac{\partial\mathcal{U}^{j+1}}{\partial x},\frac{\partial\mathcal{U}^{j+1}}{\partial x}\right)=\sum_{k=0}^{j}\mathcal{N}_{j,k}\left(\mathcal{U}^{k},\mathcal{U}^{j+1}\right).$$

Using (9), we get

$$\begin{aligned}\left\|\mathcal{U}^{j+1}\right\|_1^2 &\leq \sum_{k=0}^{j}\mathcal{N}_{j,k}\left\|\mathcal{U}^{k}\right\|_0\left\|\mathcal{U}^{j+1}\right\|_0,\\ &\leq \sum_{k=0}^{j}\mathcal{N}_{j,k}\left\|\mathcal{U}^{0}\right\|_0\left\|\mathcal{U}^{j+1}\right\|_0,\\ &\leq \left\|\mathcal{U}^{0}\right\|_0\left\|\mathcal{U}^{j+1}\right\|_1.\end{aligned}$$

Finally, we have

$$\left\|\mathcal{U}^{j+1}\right\|_1\leq\left\|\mathcal{U}^{0}\right\|_0.$$

□

Theorem 2. *Assume that (1) has a unique solution* $\mathcal{U}(t_{j+1})$ *at* $t=t_{j+1}$ *and* $\mathcal{U}^{j+1}$, $j=0,\cdots,$ $J-1$, *is the solution of this semi-discrete form in (8) with the initial condition. Then, we have the next error estimate for* $0<\gamma<1$

$$\left\|\mathcal{U}(t_{j+1})-\mathcal{U}^{j+1}\right\|_1\leq C\Delta t^2 \quad , \quad j=0,\cdots,J-1.$$

Proof. First denote the error $\varepsilon^{j+1}=\mathcal{U}(t_{j+1})-\mathcal{U}^{j+1}$ at $t=t_{j+1}$ for $j=0,\cdots,J-1$. The exact solution $\mathcal{U}(t_{j+1})$ satisfy the semi-discrete form (8), then we have

$$(\mathcal{U}(t_{j+1}),\mathcal{W})+r\left(\frac{\partial\mathcal{U}(t_{j+1})}{\partial x},\frac{\partial\mathcal{W}}{\partial x}\right)=\sum_{k=0}^{j}\mathcal{N}_{j,k}\left(\mathcal{U}(t_k),\mathcal{W}\right)-r(E_{\Delta t},\mathcal{W}). \tag{10}$$

Subtracting (8) from (10), we get

$$\left(\varepsilon^{n+1},\mathcal{W}\right)+r\left(\frac{\partial\varepsilon^{n+1}}{\partial x},\frac{\partial\mathcal{W}}{\partial x}\right)=\sum_{k=0}^{j}\mathcal{N}_{j,k}\left(\varepsilon^{k},\mathcal{W}\right)-r\left(E_{\Delta t},\mathcal{W}\right). \tag{11}$$

Now we begin the mathematical induction. For $j=0$ in (11) and $\varepsilon^0=0$, we have

$$\left(\varepsilon^{1},v\right)+r\left(\frac{\partial\varepsilon^{1}}{\partial x},\frac{\partial\mathcal{W}}{\partial x}\right)=-r\left(E_{\Delta t},\mathcal{W}\right).$$

Taking $\mathcal{W}=\varepsilon^1$, we obtain

$$\left\|\varepsilon^{1}\right\|_1\leq r\left\|E_{\Delta t}\right\|_0.$$

Using (6), we get the result

$$\left\|\mathcal{U}(t_1)-\mathcal{U}^{1}\right\|_1\leq C(\Delta t)^2.$$

Now, we suppose that

$$\left\|\mathcal{U}(t_k)-\mathcal{U}^{k}\right\|_1\leq C(\Delta t)^2 \quad , \quad \text{for } k=1,\cdots,j. \tag{12}$$

We need the prove that (12) holds for $k = j + 1$.
Taking $\mathcal{W} = \varepsilon^{j+1}$ in (11), yields

$$\begin{aligned}\left\|\varepsilon^{j+1}\right\|_1^2 &\leq \sum_{k=0}^{j}\mathcal{N}_{j,k}.\left\|\varepsilon^k\right\|_0.\left\|\varepsilon^{j+1}\right\|_0 + r\left\|E_{\Delta t}\right\|_0.\left\|\varepsilon^{j+1}\right\|_0\\ &\leq \left(\sum_{k=0}^{j}\mathcal{N}_{j,k}C(\Delta t)^2 + C(\Delta t)^2\right)\left\|\varepsilon^{j+1}\right\|_0\\ &\leq C(\Delta t)^2\left\|\varepsilon^{j+1}\right\|_1.\end{aligned}$$

Thus, we have

$$\left\|\mathcal{U}(t_{j+1}) - \mathcal{U}^{j+1}\right\|_1 \leq C(\Delta t)^2, \tag{13}$$

and this completes the proof. □

3 Space discretization: Finite element method

Let S_h denote a uniform partition of Ω which is given by

$$a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b,$$

where m is a positive integer.

Let $h = (b-a)/m = x_i - x_{i-1}$ and $\Omega_i = [x_{i-1}, x_i]$ for $i = 1, \cdots, m$. Define by χ_h the space of piecewise polynomials of order n with $n \in \mathbb{N}$ on the mesh S_h,

$$\chi_h = \left\{\mathcal{W} : \mathcal{W}|_{\Omega_i} \in P_n(\Omega_i), \mathcal{W} \in C(\Omega)\right\}.$$

Let $\mathcal{U}_h^{j+1}$ be the finite element solution at $t = t_{j+1}$, then we have the full discrete scheme of problem (1) for $0 < \gamma < 1$ given by

$$\left(\mathcal{U}_h^{j+1}, \mathcal{W}_h\right) + r\left(\frac{\partial \mathcal{U}_h^{j+1}}{\partial x}, \frac{\partial \mathcal{W}_h}{\partial x}\right) = \sum_{k=0}^{j}\mathcal{N}_{j,k}\left(\mathcal{U}_h^k, \mathcal{W}_h\right) + r\left(f^{j+1}, \mathcal{W}_h\right). \tag{14}$$

Since $\chi_h \subset H_0^1(\Omega)$ similarity to Theorem (1), we have that (14) satisfies the proprieties of the Lax-Milgram Lemma. Therefore, the existence and the uniqueness of the full discrete variational form (14) is derived.

3.1 The error estimate

Theorem 3. *Assume that problem* (1) *has a solution satisfying* $\mathcal{U}_t \in L^2(I, H^2(\Omega))) \cap L^\infty(I, H^2)$, $\mathcal{U}_{tt} \in L^2(I, L^2(\Omega))$ *such that* $\mathcal{U}^0 \in H^2(\Omega)$ *and finite element solution* (14) *is convergent to the solution of problem* (1) *on* I *as* $\Delta t,\ h \to 0$. *Then, the approximation solution satisfies,*

$$\left\|\mathcal{U}(t_{j+1}) - \mathcal{U}_h^{j+1}\right\|_1 \leq C\left[h^n\left\|\mathcal{U}\right\|_{L^\infty(H^{n+1}(\Omega))} + (\Delta t)^2\right].$$

Proof. To give an estimation of the error, we need first to discuss the error at $t = t_{j+1}$ for $j = 0, 1, \cdots, J-1$. Define $\bar{\varepsilon}^{j+1} = \mathcal{U}(t_{j+1}) - \mathcal{U}_h^{j+1}$ and for $U^{j+1} \in \chi_h$. Define $\Phi^{j+1} = \mathcal{U}(t_{j+1}) - U^{j+1}$ and $\Psi^{j+1} = U^{j+1} - \mathcal{U}_h^{j+1}$. So we get $\bar{\varepsilon}^{j+1} = \Phi^{j+1} + \Psi^{j+1}$. The exact solution at $t = t_{j+1}$ also satisfy,

$$\begin{aligned}\left(\mathcal{U}(t_{j+1}), \mathcal{W}_h\right) + r\left(\frac{\partial \mathcal{U}(t_{j+1})}{\partial x}, \frac{\partial \mathcal{W}_h}{\partial x}\right) &= \sum_{k=0}^{j}\mathcal{N}_{j,k}\left(\mathcal{U}(t_k, \mathcal{W}_h\right)\\ &+ r\left(f^{j+1}, \mathcal{W}_h\right) - r\left(E_{\Delta t}, \mathcal{W}_h\right).\end{aligned} \tag{15}$$

Subtracting (14) from (15), we get

$$\left(\bar{\varepsilon}^{j+1}, \mathcal{W}_h\right) + r\left(\frac{\partial \bar{\varepsilon}^{j+1}}{\partial x}, \frac{\partial \mathcal{W}_h}{\partial x}\right) = \sum_{k=0}^{j} \mathcal{N}_{j,k}\left(\bar{\varepsilon}^k, \mathcal{W}_h\right) - r\left(E_{\Delta t}, \mathcal{W}_h\right). \tag{16}$$

Substituting $\bar{\varepsilon}^{j+1}$ and $\mathcal{W}_h$ by $\Phi^{j+1} + \Psi^{j+1}$ and Ψ^{j+1} respectively into (16) yields

$$\begin{aligned}\left(\Psi^{j+1}, \Psi^{j+1}\right) + r\left(\frac{\partial \Psi^{j+1}}{\partial x}, \frac{\partial \Psi^{j+1}}{\partial x}\right) &= \sum_{k=0}^{j} \mathcal{N}_{j,k}\left(\bar{\varepsilon}^k, \Psi^{j+1}\right) - \left(\Phi^{j+1}, \Psi^{j+1}\right) \\ &\quad - r\left(\frac{\partial \Phi^{j+1}}{\partial x}, \frac{\partial \Psi^{j+1}}{\partial x}\right) - r\left(E_{\Delta t}, \Psi^{j+1}\right).\end{aligned}$$

Then, we have

$$\begin{aligned}\left\|\Psi^{j+1}\right\|_1^2 &\le \sum_{k=0}^{j} \mathcal{N}_{j,k}\left\|\bar{\varepsilon}^k\right\|_0 \cdot \left\|\Psi^{j+1}\right\|_0 + \left\|\Phi^{j+1}\right\|_0 \cdot \left\|\Psi^{j+1}\right\|_0 + r\left\|\Phi^{j+1}\right\|_1 \cdot \left\|\Psi^{j+1}\right\|_1 \\ &+ r\left\|E_{\Delta t}\right\|_0 \cdot \left\|\Psi^{j+1}\right\|_0.\end{aligned}$$

Using the estimation

$$\left\|\Psi^{j+1}\right\|_0 \le \left\|\Psi^{j+1}\right\|_1, \left\|\Phi^{j+1}\right\|_0 \le Ch^{n+1}\left\|\mathcal{U}(t_j)\right\|_{n+1} \text{ and } \left\|\Phi^{j+1}\right\|_1 \le Ch^n\left\|\mathcal{U}(t_j)\right\|_{n+1},$$

we obtain,

$$\begin{aligned}\left\|\Psi^{j+1}\right\|_1 &\le \sum_{k=0}^{j} \mathcal{N}_{j,k}\left\|\bar{\varepsilon}^k\right\|_0 + h^{n+1}\left\|\mathcal{U}(t_j)\right\|_{n+1} + rh^n\left\|\mathcal{U}(t_j)\right\|_{n+1} + r(\Delta t)^2 \\ &\le \sum_{k=0}^{j} \mathcal{N}_{j,k}\left\|\bar{\varepsilon}^k\right\|_0 + C\left(h^n\left\|\mathcal{U}(t_j)\right\|_{n+1} + (\Delta t)^2\right).\end{aligned}$$

From the error definition $\bar{\varepsilon}^{j+1} = \Phi^{j+1} + \Psi^{j+1}$ and the above analysis, one has

$$\begin{aligned}\left\|\bar{\varepsilon}^{j+1}\right\|_1 &\le \left\|\Phi^{j+1}\right\|_1 + \left\|\Psi^{j+1}\right\|_1 \\ &\le \sum_{k=0}^{j} \mathcal{N}_{j,k}\left\|\bar{\varepsilon}^k\right\|_0 + C\left(h^n\left\|\mathcal{U}(t_j)\right\|_{n+1} + (\Delta t)^2\right) + Ch^n\left\|\mathcal{U}(t_j)\right\|_{n+1} \\ &\le \sum_{k=0}^{j} \mathcal{N}_{j,k}\left\|\bar{\varepsilon}^k\right\|_0 + C\left(h^n\left\|\mathcal{U}(t_j)\right\|_{n+1} + (\Delta t)^2\right).\end{aligned}$$

Then, we have

$$\left\|\bar{\varepsilon}^{j+1}\right\|_1 \le \sum_{k=0}^{j} \mathcal{N}_{j,k}\left\|\bar{\varepsilon}^k\right\|_0 + C\left(h^n\left\|\mathcal{U}\right\|_{L^\infty(H^{n+1}(\Omega))} + (\Delta t)^2\right).$$

Finally, we use inductions to obtain error estimates, in a similar way as in Theorem 2, and these complete the proof. □

4 Numerical example

Here we carry out a numerical example to illustrate the effectiveness of our numerical method, for that, let S_h be the uniform classical partition of $[a, b]$ and we choose χ_h to be the space of all piecewise linear functions on S_h which means $n = 1$. Hence, χ_h can be expressed by

$$\chi_h = \left\{\mathcal{W} : \mathcal{W}|_{\Omega_i} \in P_1(\Omega_i), \mathcal{W} \in C(\Omega)\right\},$$

where $P_1(\Omega_i)$ is the space of linear polynomial on Ω_i.

Then, we can associate the test function of space χ_h by usual basis hat functions.

$$\begin{cases} {}_0^{CF}\mathcal{D}_t^{\gamma}\mathcal{U}(x,t) - \dfrac{\partial^2 \mathcal{U}(x,t)}{\partial x^2} = f(x,t), \quad 0<\gamma<1, \quad [0,1]\times[0,1] \\ \mathcal{U}(x,0) = 0 \quad , \quad x \in [0,1] \\ \mathcal{U}(0,t) = 0, \quad \mathcal{U}(1,t) = t \quad , \quad t \in [0,1], \end{cases} \tag{17}$$

where $f(x,t) = \dfrac{x^3}{\gamma}\left(1 - e^{\left[\dfrac{-\gamma t}{1-\gamma}\right]}\right) - 6tx$. Thus, here the exact solution is done by $\mathcal{U}(x,t) = tx^3$.

Tables 1 and 2 show the approximation errors and the convergence order of the finite element scheme. In Table 1, we set $h = \Delta t$ to ensure that the space discretization error is the same as the time error. In Table 2, we take $\Delta t = 0.001$, a small value enough here to check the space error and the convergence order. So, we can also check that the numerical convergence order, approaching 2, aligns consistently with the theoretical analysis.

Table 1: The error estimates and convergence order for α and $h = \Delta t$

$h = \Delta t$	$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
	error	order	error	order	error	order
1/10	$4.1179E-4$		$9.4884E-4$		$2.0152E-3$	
1/20	$1.0810E-4$	1.93	$2.751E-4$	1.79	$9.8783E-4$	1.03
1/40	$2.7711E-5$	1.96	$7.4226E-5$	1.89	$3.5160E-4$	1.49
1/80	$7.0161E-6$	1.98	$1.9287E-5$	1.94	$1.0521E-4$	1.74
1/160	$1.7652E-6$	2.00	$4.9165E-6$	1.97	$2.8795E-5$	1.87

Table 2: The error estimates and convergence order for α and $\Delta t = 0.001$

$h = \Delta t$	$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
	error	order	error	order	error	order
1/10	$4.4309E-4$		$1.2032E-3$		$7.3097E-3$	
1/20	$1.1223E-4$	1.98	$3.1035E-4$	1.95	$1.9082E-3$	1.94
1/40	$2.8232E-5$	1.99	$7.8795E-5$	1.98	$4.8731E-4$	1.97
1/80	$7.0794E-6$	2.00	$1.9851E-5$	1.99	$1.2312E-4$	1.98
1/160	$1.7725E-6$	2.00	$4.9817E-6$	1.99	$3.0943E-5$	1.99

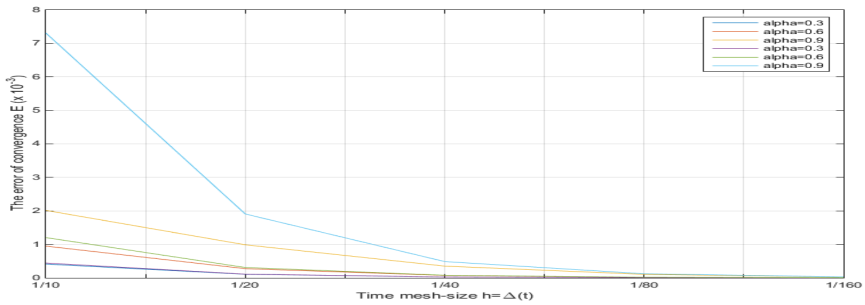

Figure 1: Convergence errors for different γ and σ

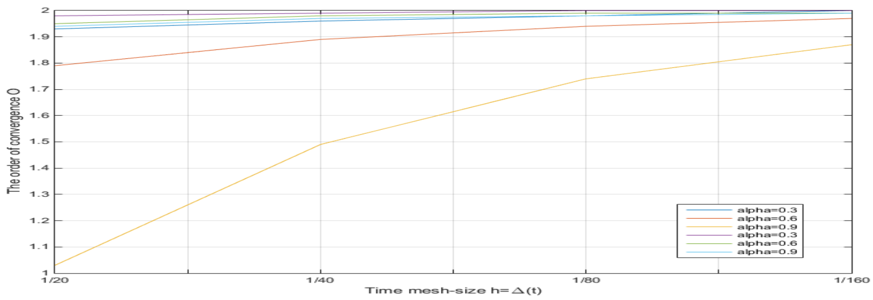

Figure 2: Convergence orders for different γ and σ

5 Conclusion

Employing the Caputo and Fabrizio fractional derivative, which addresses both temporal and spatial variables. We proposed in this work to use the finite element method for solving any fractional time partial differential equations based on the fractional order derivative. Although we discretized the fractional time derivative by using the classical finite difference scheme, which is a second-order accuracy. We applied the finite element method for the spatial derivative to approximate the space derivative and obtain directly the full discretization scheme with essentially convergence order of $O((\Delta t)^2 + h^{n+1})$. Our method proved effective through numerical testing.

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